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The Most General Planar Transformations that Map Hyperbolas to Hyperbolas

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THE MOST GENERAL PLANAR TRANSFORMATIONS THAT MAP HYPERBOLAS TO HYPERBOLAS

JAMES HAYS AND TODD MITCHELL

Abstract. The space of vertical and horizontal right hyperbolas and the lines tangent to these hyperbolas is considered in the double plane. It is proved that an injective map from the middle region of a considered hyperbola that takes hyperbolas and lines in this space to other hyperbolas and lines in this space must be a direct or indirect linear fractional transformation.

1. Introduction

It is well-known from complex analysis that Möbius transformations map circles and lines to other circles and lines in the extended complex plane, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. (See Figure 1.)

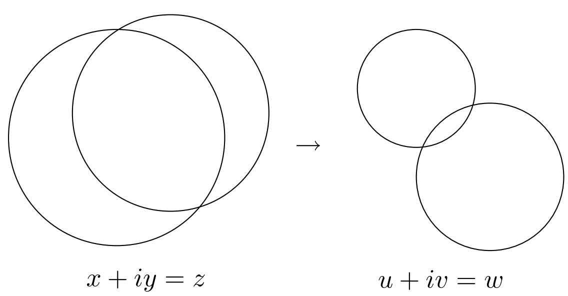

Figure 1. Möbius transformations of the complex plane transform circles into other circles

Carathéodory proved the following converse result in 1937 (reprinted in [1]):

Theorem 1 (Carathéodory, [3]). *Every arbitrary one to one correspondence between the points of a circular disc C and a bounded set C' by which circles lying completely in C are transformed into circles lying in C' must always be either a direct or inverse transformation of Möbius.*

In other words, the Möbius transformations are the *only* transformations that map circles and lines to other circles and lines. In this article we consider the analogous problems for the double numbers, also known as the perplex numbers. We represent the extended double plane as $\hat{\mathbb{P}} = \mathbb{P} \cup H_\infty$, where $\mathbb{P} = \{z = x + ky \mid$

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$x, y \in \mathbb{R},\ k^2 = 1\}$ and $H_\infty = \{(\alpha \pm \alpha k)^{-1} \mid \alpha \in \mathbb{R} \cup \{\infty\}\}$. It is not hard to show that the linear fractional transformations of the double plane map vertical and horizontal right hyperbolas and lines in $S = \{y = mx + b \mid m, b \in \mathbb{R} \mid m \neq \pm 1\}$ to other vertical and horizontal right hyperbolas and lines in S. (See Figure 2.)

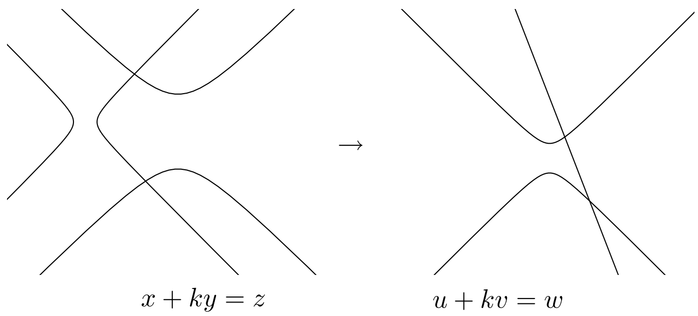

FIGURE 2. Linear fractional transformations of the double plane transform hyperbolas into other hyperbolas

We prove the following converse theorem:

Theorem 2. *Every injective map from a closed middle region bounded by a horizontal or vertical right hyperbola that maps horizontal and vertical right hyperbolas and the lines in $S = \{y = mx + b \mid m, b \in \mathbb{R} \mid m \neq \pm 1\}$ to horizontal and vertical right hyperbolas and lines in S is a direct or indirect linear fractional transformation.*

The space we consider contains only those hyperbolas with asymptotes that have slopes $+1$ and -1. Of course all of these hyperbolas have vertical and horizontal lines of symmetry. However, we classify vertical right hyperbolas as those with a vertical line of symmetry that intersects the hyperbola, and horizontal right hyperbolas as those with a horizontal line of symmetry that intersects the hyperbola. The space also consists of all lines except those with slope ± 1. We take a middle region to be that which is between both branches of a hyperbola, as shown in Figure 3.

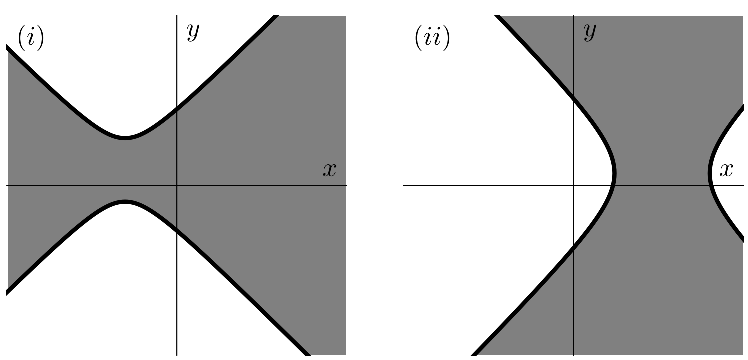

FIGURE 3. Middle regions of a vertical (i) and horizontal (ii) hyperbola

We mention that in the dual plane, $\mathbb{D} = \{z = x + jy \mid x, y \in \mathbb{R} \mid j^2 = 0\}$, the linear fractional transformations are Laguerre transformations, and that they take vertical parabolas and nonvertical lines to other vertical parabolas and nonvertical lines. Recently, Bolt, Ferdinands and Kavlie proved the similar converse result.

Theorem 3 (Bolt, Ferdinands, and Kavlie [2]). *Every injective map from a closed region bounded by a vertical parabola or nonvertical line that maps vertical parabolas and nonvertical lines to vertical parabolas and nonvertical lines is the composition of a non-isotropic dilation $d_\lambda : (x, y) \to (\lambda x, \lambda^2 y)$, $0 \neq \lambda \in \mathbb{R}$, with a direct or indirect Laguerre transformation.*

2. Geometry in the Extended Double Plane

This section is a summary of the properties and geometry associated with the double numbers. A comprehensive account on double numbers is provided by Yaglom [5].

A double number $z \in \mathbb{P}$ is a formal expression $z = x + ky$ where $x, y \in \mathbb{R}$ and $k^2 = 1$. The double numbers can be identified with the points of the real plane via $x + ky \in \mathbb{P} \leftrightarrow (x, y) \in \mathbb{R}^2$. The coordinates of $z = x + ky$ are the real component and double component, respectively. The numbers form a commutative algebra over $\mathbb{R}$ where addition and multiplication are done as usual. For instance, $(3 - 7k) + (4 + 2k) = (3 + 4) + (-7 + 2)k = 7 - 5k$ and $(3 - 7k) \cdot (4 + 2k) = (3 \cdot 4 - 7 \cdot 2) + (3 \cdot 2 - 7 \cdot 4)k = -2 - 22k$.

Figure 4 illustrates how this works in the geometry of the double plane. Addition in the double plane can be represented by the addition of position vectors. Multiplication can be represented using directed taxicab distances from points to the lines $y = x$ and $y = -x$. The taxicab distances of points to the right of the line $y = x$ and above the line $y = -x$ can be thought of as positive, whereas points to the left of the line $y = x$ and below the line $y = -x$ can be thought of as negative. To multiply two points, multiply together their directed taxicab distances from the line $y = x$ to find the new directed taxicab distance from $y = x$. Do the same for $y = -x$. These two values determine a unique point, the same one found by doing the usual algebraic computation.

Direct and indirect linear fractional transformations are those of the forms

$$\mu(z) = \frac{az + b}{cz + d} \quad \text{and} \quad \mu(z) = \frac{a\bar{z} + b}{c\bar{z} + d},$$

respectively, where $a, b, c, d \in \mathbb{P}$ and $ad - bc = \pm 1$. The condition $ad - bc = \pm 1$ is used for normalizing and does not affect the transformation. However, it is necessary that $ad - bc \neq \alpha \pm \alpha k$, where $\alpha \in \mathbb{R}$, or else μ maps $\hat{\mathbb{P}}$ to a line or to a point.

The linear fractional transformations form a group under composition. There are three types of transformations that generate the direct linear fractional transformations:

- translation: $\mu(z) = z + b$ for $b \in \mathbb{P}$
- rotation and dilation: $\mu(z) = az$ for $a \in \mathbb{P}$, $a \neq \alpha \pm \alpha k$, where $\alpha \in \mathbb{R}$
- inversion: $\mu(z) = 1/z$.

The full group is obtained by including conjugation. Both the direct and indirect transformations preserve a specific space of hyperbolas and lines. The space under

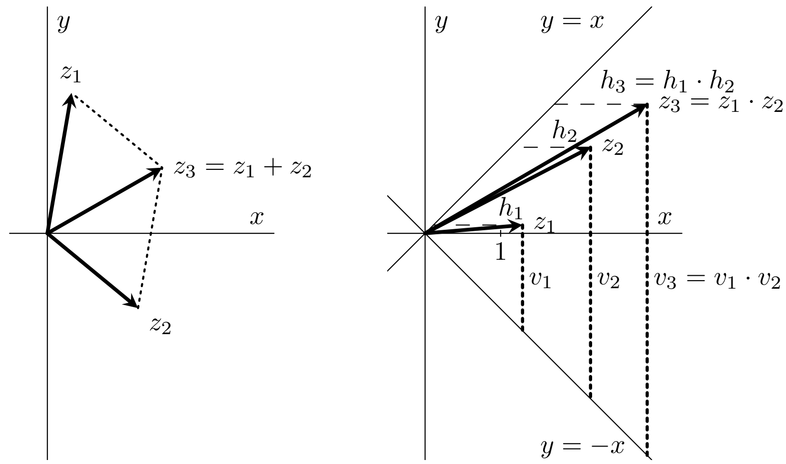

FIGURE 4. Addition and multiplication of double numbers

consideration can be described by the equation

$$Az\bar{z} + \text{Real}((B + Ck)z) + D = 0$$

where $4AD + C^2 - B^2 \neq 0$ and $A, B, C, D \in \mathbb{R}$. One can verify the preservation of this space for the direct transformations by showing that each of the three transformations which compose linear fractional transformations preserve the space. This is also true for the indirect transformations because the conjugation $z \to \bar{z} = x - ky$ also preserves the space.

By using stereographic projection, the extended double plane $\hat{\mathbb{P}} = \mathbb{P} \cup H_\infty$ can be viewed as an infinite hyperboloid as shown in Figure 5. We found it useful to consider the hyperboloid $x^2 - y^2 + (z-1)^2 = 1$, where the xy-plane is the double plane, using the point $(0, 0, 2)$ as a projection point. Drawing a line through any point in the double plane and the projection point yields a line which intersects the hyperboloid in a second location; this is the associated point on the hyperboloid. Hyperbolas and lines in the double plane correspond with planar cross-sections of the hyperboloid that take the form of ellipses and hyperbolas. A second hyperboloid $-x^2 + y^2 + (z-1)^2 = 1$ can be used to compactify those hyperbolas in the double plane which did not correspond with ellipses on the first hyperboloid.

The set $H_\infty = \{(\alpha \pm \alpha k)^{-1} \mid \alpha \in \mathbb{R} \cup \{\infty\}\}$ corresponds with two lines at infinity that intersect at the point $(0 + 0k)^{-1}$. For a given hyperbola $[A, B, C, D]$, it can be verified what happens to the points under each of the three transformations that compose the linear fractional transformations. It can also be verified which two points at H_∞ each hyperbola or line intersects:

i) $(\alpha_1 + \alpha_1 k)^{-1}$ where $\alpha_1 = \begin{cases} \frac{-A}{B-C} & B \neq C \\ \infty & B = C \end{cases}$

ii) $(\alpha_2 - \alpha_2 k)^{-1}$ where $\alpha_2 = \begin{cases} \frac{-A}{B+C} & B \neq -C \\ \infty & B = -C \end{cases}$

The lines have $A = 0$, thus they can be considered to intersect $(0 + 0k)^{-1}$ twice.

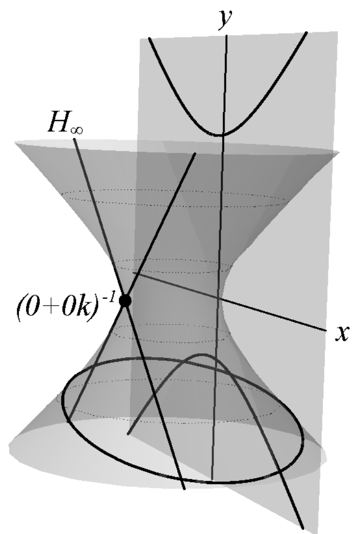

FIGURE 5. Representation of a hyperbola on the hyperboloid projected onto the double plane

3. Proof of Theorem 2

We model this proof on Carathéodory's while contriving similar arguments to those of Bolt, Ferdinands and Kavlie. We shall examine an injective map $T : P \to P'$ that maps horizontal and vertical right hyperbolas and the lines in S that are contained in a middle region P to other horizontal and vertical right hyperbolas and lines in S that are contained in P'. By precomposing T with a linear transformation we may assume that P is the middle region of the preferred hyperbola $x^2-y^2+1=0$, that is to say $P = \{(x, y) \mid -\sqrt{x^2+1} \le y \le \sqrt{x^2+1}\}$.

3.1. **Preliminary remark.** Injectivity implies the preservation of the number of intersection points of two horizontal or vertical right hyperbolas or lines in S that are contained within the region. A point that lies on two hyperbolas h_0 and h_1 in P must lie on $T(h_0)$ and $T(h_1)$ in P'. It is clear that distinct points which lie on two hyperbolas in P may not be injectively mapped to the same point in P'. Thus intersection points are not lost. Additionally, if there is a point z such that $T(z)$ lies on $T(h_0)$ and $T(h_1)$, then it must be the case that z lay on both h_0 and h_1, meaning intersection points cannot be created. Specifically, non-intersecting hyperbolas remain non-intersecting, tangent hyperbolas remain tangent, and twice-intersecting hyperbolas remain twice-intersecting.

3.2. **First Normalization.** Through post-composing T with several linear fractional transformations we may assume that

(1) $T : \{x^2 - y^2 + 1 = 0\} \to \{x^2 - y^2 + 1 = 0\}$.
(2) $T(0, 1) = (0, 1)$
(3) $T((0 + 0k)^{-1}) = (0 + 0k)^{-1}$

Suppose the original transformation is T_0, then we will begin by composing it with a transformation, so that the composed transformation will take the preferred hyperbola, $x^2 - y^2 + 1 = 0$, to itself and at the same time map the point $(0, 1)$ to $(0, 1)$. Because three points are enough to determine a hyperbola of the considered form, we need only to specify that any three points from $x^2 - y^2 + 1 = 0$ are mapped back onto that hyperbola. To satisfy condition (2) at the same time we choose one of these three points to be $(0, 1)$ and map it to $(0, 1)$.

To aid in specifying the necessary transformation, we shall compose two transformations. The first transformation will take the three points to the real line, and the second is the inverse of a transformation that takes the images of the three points to the real line. Call $T_0(0, 1) = w_0$, then choose two other points z_1 and z_2 on the preferred hyperbola and call $T_0(z_1) = w_1$, and $T_0(z_2) = w_2$. To map w_0, w_1, and w_2 to the real line at $(0, 0)$, $(1, 0)$, and $(0 + 0k)^{-1}$ respectively, we use

$$\mu_0(z) = \frac{(z - w_0)(w_1 - w_2)}{(z - w_2)(w_1 - w_0)}.$$

Similarly, we map $(0, 1)$, z_1, and z_2 to the real line at $(0, 0)$, $(1, 0)$, and $(0 + 0k)^{-1}$ using

$$\mu_1(z) = \frac{(z - k)(z_1 - z_2)}{(z - z_2)(z_1 - k)}.$$

Then $T_1 \stackrel{\text{def}}{=} \mu_1^{-1} \circ \mu_0 \circ T_0$ gives us the first and second conditions. In achieving the third condition we must be careful not to disturb the first two conditions which we have already established. Maintaining condition (1) limits the available transformations to the forms

$$\mu'(z) = \frac{dz + b}{-\bar{b}z + \bar{d}} \quad \text{and} \quad \mu''(z) = \frac{d\bar{z} + b}{-\bar{b}\bar{z} + \bar{d}}.$$

Maintaining condition (2) further limits the available transformations by requiring $b + \bar{b} = -k(d - \bar{d})$. This leaves us with the $T_1((0 + 0k)^{-1}) = w_3 = x_3 + ky_3$.

To complete the first normalization we set $T_2 \stackrel{\text{def}}{=} \mu' \circ T_1$ where b and d are chosen in the following manner. If $x_3 = 0$ choose $b = -k$, and $d = y_3$. If $x_3 \neq 0$ choose $b = 1 + k(y_3 - 1)/x_3$, and $d = x_3 + (y_3 - {y_3}^2)/x_3 - k$. (A similar argument works for μ''.) At this stage conditions (1), (2), and (3) have been met, and the only remaining linear fractional transformations which maintain all three conditions are $\mu(z) = z$ and $\mu(z) = -\bar{z}$.

3.3. **Parallel Lines.** As you will recall from section §2, the hyperbolas which contain $(0+0k)^{-1}$ are precisely the lines. Since $(0+0k)^{-1}$ has been preserved, it follows that under T lines are mapped to lines. Remembering the preliminary remark on the preservation of intersection points (§3.1), lines intersecting only at $(0 + 0k)^{-1}$ (parallel lines) must continue to intersect at $(0 + 0k)^{-1}$ and nowhere else. Thus parallel lines go to parallel lines.

3.4. **Three Horizontal Lines.** Because lines go to lines and because the point $(0, 1)$ is preserved, $T(\{y = 1\})$ is a line of the form $y = mx + 1$ (one containing the point $(0, 1)$). Keeping in mind that the line $y = 1$ is tangent to $x^2 - y^2 + 1 = 0$, it must be that $T(\{y = 1\})$ is also tangent to $x^2 - y^2 + 1 = 0$. The only line of the form $y = mx + 1$ with this property arises by setting $m = 0$. Thus the line $y = 1$ is preserved. By the previous section on parallel lines, this indicates that horizontal lines go to horizontal lines. The line $y = -1$ must go to a horizontal line tangent to $x^2 - y^2 + 1 = 0$. There are only two possibilities, $y = 1$ and $y = -1$, and injectivity forces the latter.

Now consider the line $y = 0$. Aside from the infinity point and the origin, all of the remaining points on this line have exactly two tangent lines to $x^2 - y^2 + 1 = 0$, and thus their images must possess two tangents to its image. Note that when we say a point not on a hyperbola has a tangent line to that hyperbola, we mean that there exists a point on the hyperbola so that the tangent line at this point also goes through the point not on the hyperbola. The infinity point is preserved, leaving only one finite point that is permitted fewer than two tangents to $x^2 - y^2 + 1 = 0$ in the image. Note that no point on the asymptotes, $y = \pm x$, has two tangent lines to the preferred hyperbola. Thus $T(\{y = 0\})$ is a horizontal line permitted to intersect the asymptotes at most once, forcing the line to be preserved. Also note that these tangency properties ensure that if any point on the line $y = 0$ other than the origin were mapped to the origin it would lose tangent lines, thus the origin is preserved.

3.5. **Second Normalization.** We continue observing the number of tangent lines to the preferred hyperbola, now considering points on the line $y = 1$. Since the set of points $\{(1, 1), (-1, 1)\}$ lies on the asymptotes of the hyperbola, both points in the set have but one tangent to $x^2 - y^2 + 1 = 0$. As every other point on the line $y = 1$ has two lines of tangency to the hyperbola, only the set of points $\{(1, 1), (-1, 1)\}$ may be mapped to $\{(1, 1), (-1, 1)\}$. If $T_2(1, 1) = (-1, 1)$ then we say $T \stackrel{\text{def}}{=} \mu_3 \circ T_2$ where $\mu_3(z) = -\bar{z}$. Otherwise we say $T \stackrel{\text{def}}{=} T_2$. This preserves the points $(1, 1)$ and $(-1, 1)$ by specifying the last of our linear fractional transformations.

As a consequence of this, we can show that T maps the left half of the x-axis to itself. To see this, notice that there exists a line through $(1, 1)$ and any point $(x, 0)$ if and only if $x < 0$. Since the line $y = 0$ is preserved and lines through (1,1) must go to lines through (1,1), then $T(\{(x, 0) \mid x < 0\}) \subseteq \{(x, 0) \mid x < 0\}$. By a similar argument T takes the right half of the x-axis to itself.

3.6. **Zig-Zag.** By an argument similar to that for the set of points $\{(1, 1), (-1, 1)\}$, the set of points $\{(1, -1), (-1, -1)\}$ is also mapped to itself. We build off of the previous normalization to determine which point is mapped to $(1, -1)$. The argument is illustrated in Figure 6.

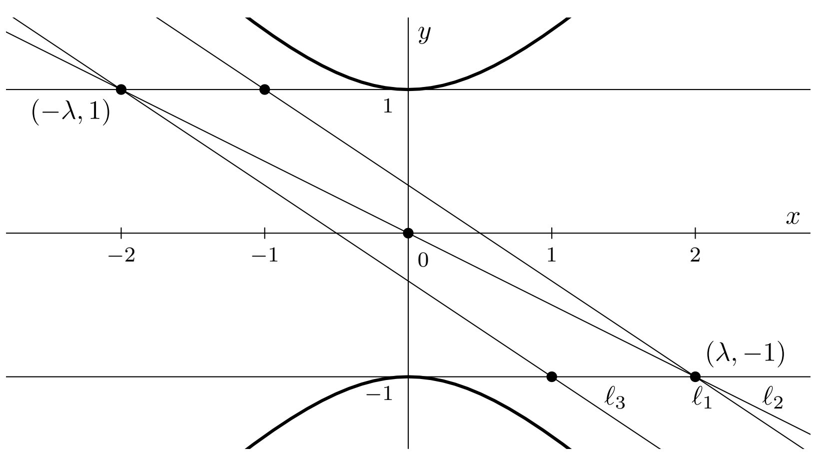

FIGURE 6. Pre-image for the preservation of (1,-1)

We choose a point $(\lambda, -1)$ with $\lambda > 1$ for some $\lambda \in \mathbb{R}$. Because the line $y = -1$ is preserved, this point goes to $(\Lambda, -1)$ for some $\Lambda \in \mathbb{R}$. Let ℓ_1 be the line containing $(-1, 1)$ and $(\lambda, -1)$. In order for $T(\ell_1)$ to be a line containing $(-1, 1)$ and $(\Lambda, -1)$, and not intersect $x^2 - y^2 + 1 = 0$, it is required that $\Lambda > 1$. Let ℓ_2 be the line through $(\lambda, -1)$ and $(0, 0)$, which then intersects the line $y = 1$ at $(-\lambda, 1)$. Since the origin is preserved, $T(\ell_2)$ intersects the line $y = 1$ at $(-\Lambda, 1)$. Let ℓ_3 be the line through $(-\lambda, 1)$ that is parallel to ℓ_1. Then ℓ_3 intersects the line $y = -1$ at the point $(1, -1)$. $T(\ell_3)$ must be parallel to $T(\ell_1)$, and thus also intersects the line $y = -1$ at $(1, -1)$. Therefore $(1, -1)$ is preserved, and by injectivity this simultaneously forces the other member of the set $\{(1, -1), (-1, -1)\}$ to be preserved.

3.7. **A Specific Case.** Next we preserve a third point on the preferred hyperbola by constructing lines and hyperbolas off of the specific point (2,0) as illustrated in Figure 7. Recall from §3.5 that the right half of the x-axis is preserved. This implies that $T(2, 0) = (\gamma, 0)$ with $\gamma > 0$. We draw the hyperbola $x^2 - y^2 - 4 = 0$ through $(2, 0)$, noting that the image of this hyperbola is $x^2 - y^2 - \gamma^2 = 0$. (The image can't be a line and the only remaining hyperbola through $(\gamma, 0)$ that does not intersect $x^2 - y^2 + 1 = 0$ is $x^2 - y^2 - \gamma^2 = 0$.) The hyperbola $x^2 - y^2 - 4 = 0$ intersects the left half of the x-axis at $(-2, 0)$ and $x^2 - y^2 - \gamma^2 = 0$ intersects the left half of the x-axis at $(-\gamma, 0)$. This implies $T(-2, 0) = (-\gamma, 0)$.

We now draw in two lines, one through the points $(2, 0)$ and $(-1, 1)$ and the other through the points $(-2, 0)$ and $(1, 1)$. The intersection point of these two lines is $(0, 2/3)$. The images of these two lines intersect at $(0, \gamma/(\gamma + 1))$ which implies $T(0, 2/3) = (0, \gamma/(\gamma + 1))$. Next we examine the point $(-\sqrt{5}, -1)$, one of the intersections of the hyperbola $x^2 - y^2 = 4$ and the line $y = -1$, (the other being $(\sqrt{5}, -1)$). Notice that there is a line through this point and the point $(1, 1)$ which does not intersect the preferred hyperbola, but that there is no such line for the point $(\sqrt{5}, -1)$. The corresponding intersection points after the transformation are $(-\sqrt{\gamma^2 + 1}, -1)$ and $(\sqrt{\gamma^2 + 1}, -1)$. There is a line through $(-\sqrt{\gamma^2 + 1}, -1)$ and $(1, 1)$ that does not intersect the preferred hyperbola, but there is no such line for the other solution. These observations force $T(-\sqrt{5}, -1) = (-\sqrt{\gamma^2 + 1}, -1)$. If we examine the line through $(-\sqrt{5}, -1)$ and $(0, \frac{2}{3})$, we notice that it is tangent to $x^2 - y^2 + 1 = 0$ at $(\sqrt{5}/2, 3/2)$. Thus the line through $(\sqrt{\gamma^2 + 1}, -1)$ and $(0, \gamma/(\gamma + 1))$ must also be tangent to $x^2 - y^2 + 1 = 0$. This line is

$$y = \frac{1 + 2\gamma}{(\gamma + 1)\sqrt{\gamma^2 + 1}} x + \frac{\gamma}{\gamma + 1}.$$

Substituting this expression for y into $x^2 - y^2 + 1 = 0$ yields a quadratic for x in terms of γ which represents the x-coordinates of any intersection points of this line and the preferred hyperbola. Only when the two roots of the quadratic are identical is the line tangent. Equating the two roots for x gives an equation for γ which has nine solutions: double roots at $\pm i$ and -1, and single roots at $-1/2$, 0, and 2.

Due to the earlier condition that $\gamma > 0$, it is clear that $\gamma = 2$. This means that the point $(2, 0)$ is preserved, as well as all the hyperbolas and intersection points we have constructed off of it. In particular this includes $(\sqrt{5}/2, 3/2)$ and $(0, 2/3)$. Because T preserves the point $(0, 2/3)$ and maps hyperbolas to hyperbolas, it follows that the hyperbola $x^2 - y^2 + 4/9 = 0$ is preserved.

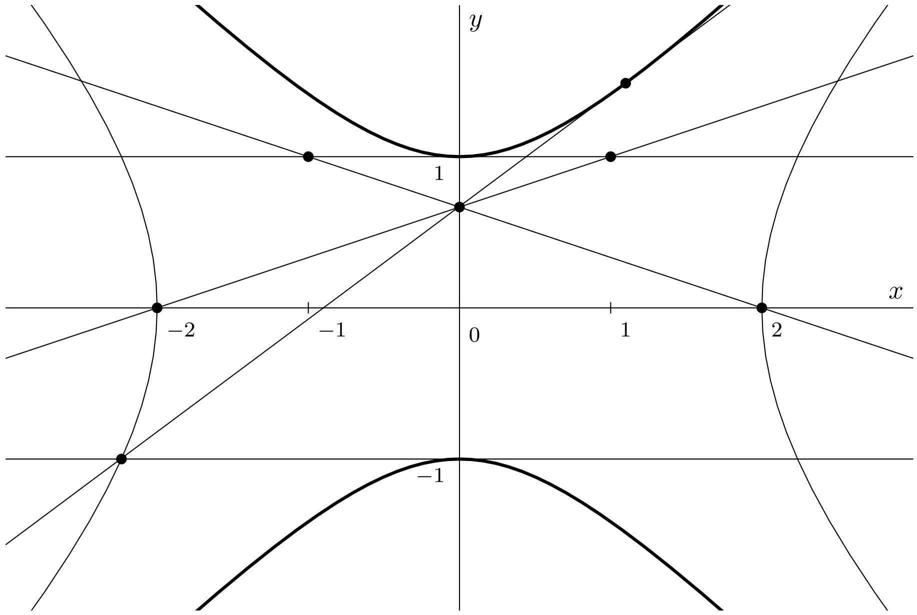

FIGURE 7. A Specific Case

3.8. **A Special Unbounded Polygon.** In this section we prove by induction that there is an entire sequence of points $\{U_0, U_1, U_2, U_3, \dots\}$ on the hyperbola $x^2-y^2+1=0$ that is preserved by the transformation. This is done by constructing a special unbounded polygon that circumscribes the hyperbola $x^2-y^2+1=0$ and inscribes the hyperbola $x^2-y^2+\operatorname{sech}^2(t_1/2)=0$ as shown in Figure 8. As we have just seen, the point $(\sqrt{5}/2, 3/2)$ on the hyperbola $x^2-y^2+1=0$, which we will refer to here as the upper hyperbola, is preserved. We shall now label this point $U_1 = (\sinh t_1, \cosh t_1)$. The fact that U_1 is on the preferred hyperbola is the base case of the induction. The tangent line through U_1 on the hyperbola is $\mathcal{T}_1 : y = \tanh(t_1)x + \operatorname{sech}(t_1)$. $\mathcal{T}_1$ intersects the line $y = 1$ at the point $L_0 = (\tanh(t_1/2), 1)$. Solving for the hyperbola of the form $x^2-y^2+d=0$ gives the lower hyperbola, $x^2-y^2+\operatorname{sech}^2(t_1/2)=0$, which passes through L_0.

$\mathcal{T}_1$ also intersects the lower hyperbola at the point

$$L_1 = (\sinh(t_1) + \cosh(t_1)(\tanh(t_1/2), 2\cosh(t_1) - 1).$$

There is a tangent line $\mathcal{T}_2$ to the upper hyperbola through L_1 which intersects the upper hyperbola at a point $U_2 = (\sinh(t_2), \cosh(t_2))$. This pattern can be repeated and each of the points which are the intersection of the tangent lines and the upper hyperbola are preserved. We show that $t_k - t_{k-1} = t_1$.

Two arbitrary consecutive points constructed in this fashion can be expressed as $U_k = (\sinh(t_k), \cosh(t_k))$ and $U_{k-1} = (\sinh(t_{k-1}), \cosh(t_{k-1}))$, choosing $t_k > t_{k-1}$. The tangent lines at these points are $\mathcal{T}_k : y = \tanh(t_k)x + \operatorname{sech}(t_k)$ and $\mathcal{T}_{k-1} : y = \tanh(t_{k-1})x + \operatorname{sech}(t_{k-1})$.

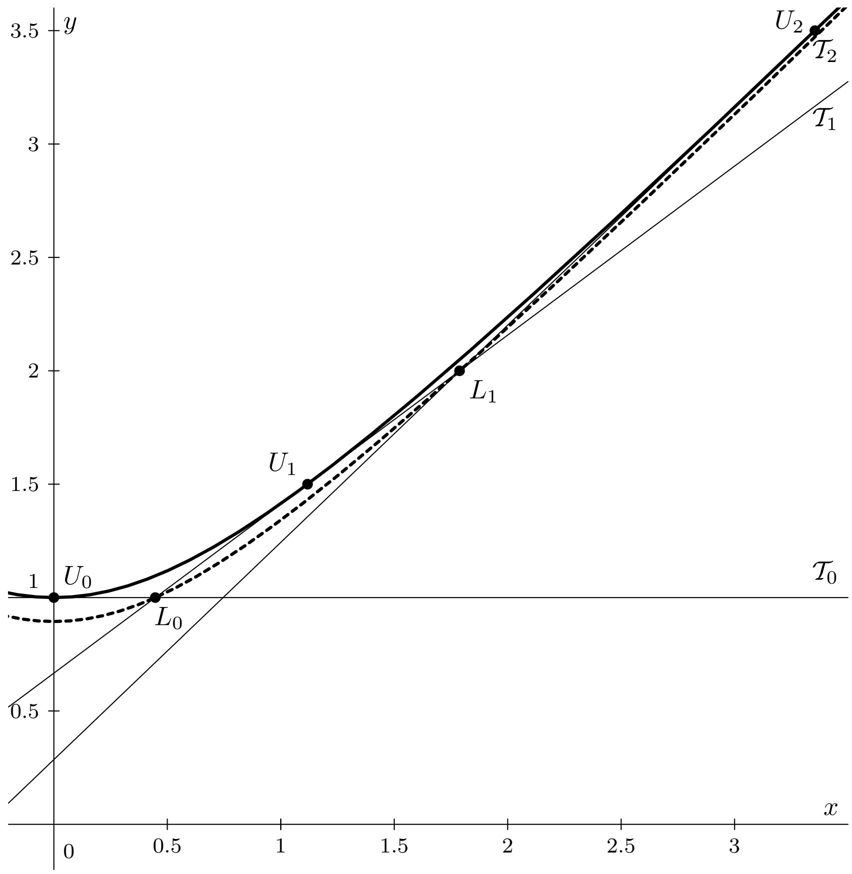

FIGURE 8. A Special Unbounded Polygon

Next we solve for the intersection of these tangent lines; the intersection is expected to be a point on the lower hyperbola. We find that the point is

$$\left(\frac{\operatorname{sech}(t_{k-1})-\operatorname{sech}(t_k)}{\tanh(t_k)-\tanh(t_{k-1})}, \tanh(t_k)\frac{\operatorname{sech}(t_{k-1})-\operatorname{sech}(t_k)}{\tanh(t_k)-\tanh(t_{k-1})}+\operatorname{sech}(t_k)\right).$$

These values for x and y can be substituted into the equation of the lower hyperbola giving

$$\left(\frac{\operatorname{sech} t_{k-1}-\operatorname{sech} t_k}{\tanh t_k-\tanh t_{k-1}}\right)^2-\left(\frac{\tanh t_k \operatorname{sech} t_{k-1}-\tanh t_{k-1}\operatorname{sech} t_k}{\tanh t_k-\tanh t_{k-1}}\right)^2+\operatorname{sech}\left(\frac{t_1}{2}\right)^2=0.$$

After some work, this simplifies to

$$\operatorname{sech}\left(\frac{t_k-t_{k-1}}{2}\right)^2=\operatorname{sech}\left(\frac{t_1}{2}\right)^2 \quad\Rightarrow\quad t_k-t_{k-1}=\pm t_1.$$

But since $t_k > t_{k-1}$ we have $t_k - t_{k-1} = t_1$. Therefore, by induction, $kt_1 = t_k$. This means that all points of the form $(\sinh(pt_1), \cosh(pt_1))$ where $p \in \mathbb{Z}^+$ are preserved. Because of the hyperbola's vertical axis of symmetry, a similar argument yields that points of the form $(\sinh(pt_1), \cosh(pt_1))$, where $p \in \mathbb{Z}$, are preserved. Finally, because of the hyperbola's horizontal axis of symmetry, one more argument yields that points of the form $(\sinh(pt_1), -\cosh(pt_1))$, where $p \in \mathbb{Z}$, are also preserved.

3.9. **Halving.** We now show that given an arbitrary preserved point on the preferred hyperbola $(\sinh(t_k), \cosh(t_k))$, the point $(\sinh(t_k/2), \cosh(t_k/2))$ is also preserved. The tangent line for a given point t_k that has been preserved is $y = \tanh(t_k)x + \operatorname{sech}(t_k)$. This tangent line intersects the line $y = -1$ at the point $(-\coth(t_k/2), -1)$. This point is preserved because tangent lines and the line $y = -1$ are preserved. There is a horizontal hyperbola through this point of the form $x^2 - y^2 + D = 0$ which is also preserved. Substituting in the x and y we find $(-\coth(t_k/2))^2 - (-1)^2 + D = 0$ and simplifying gives $D = -\operatorname{csch}^2(t_k/2)$.

The hyperbola $x^2 - y^2 - \operatorname{csch}^2(t_k/2) = 0$ crosses the x-axis at $(\pm\operatorname{csch}(t_k/2), 0)$. Since the line $y = 0$ is preserved, and since T preserves the left half of the x-axis, the point $(-\operatorname{csch}(t_k/2), 0)$ must be preserved. There is a tangent line to the preferred hyperbola through $(-\operatorname{csch}(t_k/2), 0)$ with the equation of $y = \tanh{(t_m)}x + \operatorname{sech} t_m$. Substituting the coordinates of this point into the equation and solving for t_m gives $t_m = t_k/2$. So the tangent line to the preferred hyperbola intersects it at the point $(\sinh(t_k/2), \cosh(t_k/2))$ and this point is preserved.

Using the method of this section in conjunction with the method from Section 3.8, it follows that all points of the form $(\sinh(t_1 \cdot p2^{-q}), (\cosh(t_1 \cdot p2^{-q}))$ where $p, q \in \mathbb{Z}$ are preserved. The symmetric properties of the hyperbola allow that similar arguments would preserve reflections of these points over the x-axis and y-axis. These points form a dense subset of the preferred hyperbola $x^2 - y^2 + 1 = 0$.

3.10. **A dense subset of the region.** We now show that T preserves a set of points that is dense in the region. We do so by taking all lines tangent to the hyperbola $x^2 - y^2 + 1 = 0$ at the points $(\sinh{(t_1 \cdot p2^{-q})}, \cosh{(t_1 \cdot p2^{-q})})$ where $p, q \in \mathbb{Z}$. By the previous two subsections T preserves these points of tangency. T also preserves $x^2 - y^2 + 1 = 0$, so it also preserves the lines tangent to $x^2 - y^2 + 1 = 0$ at these points. It follows that T preserves the intersection points of these tangent lines, and these intersection points form a dense subset of the region.

3.11. **Preservation of $x^2 - y^2 + 1 = 0$.** We now show that T preserves each point of the hyperbola $x^2 - y^2 + 1 = 0$. Suppose that T does not. Then there exist b and τ with $b \neq p \cdot t_1 \cdot 2^{-q}$ and $\tau \notin \{0, 1\}$ such that $T(\sinh(b), \cosh(b)) = (\sinh(\tau b), \cosh(\tau b))$. By replaying the arguments from §3.8 and §3.9 it follows that $T(\sinh(d \cdot b), \cosh(d \cdot b)) = (\sinh(d \cdot \tau b), \cosh(d \cdot \tau b))$ for $d = p \cdot 2^{-q}$. Furthermore, T would map the line tangent to $x^2 - y^2 + 1 = 0$ at $x = \sinh(d \cdot b)$ to the line tangent to $x^2 - y^2 + 1 = 0$ at $x = \sinh(d \cdot \tau b)$.

The argument from §3.10 determines how T would act on the set of points that arise as the intersections of the tangent lines to $x^2 - y^2 + 1 = 0$ for $x = \sinh(d \cdot b)$. Let us now consider some $d = p \cdot 2^{-q}$ that is fixed but arbitrary. The lines tangent to $x^2 - y^2 + 1 = 0$ at $x = \sinh(d \cdot b)$ and $x = \sinh(-d \cdot b)$ intersect at the point $(0, \operatorname{sech}(d \cdot b))$. These tangent lines are sent to the lines tangent to $x^2 - y^2 + 1 = 0$ at $x = \sinh(d \cdot \tau b)$ and $x = \sinh(-d \cdot \tau b)$, which intersect at the point $(0, \operatorname{sech}(d \cdot \tau b))$. So T maps $(0, \operatorname{sech}(d \cdot b))$ to $(0, \operatorname{sech}(d \cdot \tau b))$. Since T also maps horizontal lines to horizontal lines, it follows that T maps the horizontal line $y = \operatorname{sech}(d \cdot b)$ to $y = \operatorname{sech}(d \cdot \tau b)$.

However, a contradiction arises when $\tau > 1$ if we choose $d = p \cdot 2^{-q}$ such that $\operatorname{sech}(d \cdot \tau b) < 2/3 < \operatorname{sech}(d \cdot b)$. Under these conditions the hyperbola $x^2 - y^2 + 4/9 =$

0 intersects the line $y = \operatorname{sech}(d \cdot b)$ twice, but after the action of T, the hyperbola $x^2 - y^2 + 4/9 = 0$ does not intersect the line $y = \operatorname{sech}(d \cdot \tau b)$ even once. This contradicts the preliminary remark on injectivity (§3.1).

If $0 < \tau < 1$, choose $d = p \cdot 2^{-q}$ such that $\operatorname{sech}(d \cdot b) < 2/3 < \operatorname{sech}(d \cdot \tau b)$. Then the hyperbola $x^2 - y^2 + 4/9 = 0$ does not intersect the line $y = \operatorname{sech}(d \cdot b)$, but after the action of T, the hyperbola $x^2 - y^2 + 4/9 = 0$ intersects the line $y = \operatorname{sech}(d \cdot \tau b)$ twice, which again contradictions the properties of injectivity described in §3.1.

The case $\tau < 0$ still requires examination. Let us look at the tangent line through the point $(\sinh b, \cosh b)$. If $b > 0$, this tangent line intersects $y = 0$ at a point where $x < 0$. Meanwhile, the tangent line at $(\sinh \tau b, \cosh \tau b)$ would intersect $y = 0$ at a point where $x > 0$. This would contradict the fact that T preserves the left half of the x-axis. The argument is similar in case $b < 0$.

3.12. **Completion of the Proof.** We conclude that T preserves each point of $x^2 - y^2 + 1 = 0$ and it follows from the preservation of a dense subset of the region in §3.10 that T preserves all points in the region except the asymptotes of $x^2 - y^2 + 1 = 0$, specifically the lines $y = x$ and $y = -x$. While all other points in the region can be obtained by the intersection of tangent lines to $x^2 - y^2 + 1 = 0$, The points on $y = x$ and $y = -x$ can not.

However, for any given point on the asymptotes, there are an infinite number of lines in the region through the point. Choose any two distinct lines through the point. Since the rest of the region is preserved, the lines are preserved, and the intersection of these lines is preserved. Thus the asymptotes are preserved as well. Therefore, T acts identically on the entire region and its boundary, completing the proof of Theorem 2.

4. Further Questions

In our theorem we have only dealt with those initial regions which take the form of closed middle regions of hyperbolas. However, we wonder if there is a more natural way to set up the problem by considering the extended double plane as the compactified single-sheeted hyperboloid. If so, then this formulation could be used to extend our results.

Additionally, when considered in conjunction with the recent proofs of the analogous theorems for the dual and complex cases, this proof suggests a second follow-up question. Is there an elegant way of combining these cases into one overarching proof dealing with all three in the same way? This would not be trivial, as there are differences even in the results (recall that in the dual case non-isotropic dilations also take parabolas to parabolas). Some similar problems have already been approached in this unified way, see for instance Kisil [4]. Such a proof would certainly help to further unify these number systems.

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